Fast Decoders for Topological Quantum Codes

Guillaume Duclos-Cianci$^1$    David Poulin$^1$

$^1$Département de Physique, Université de Sherbrooke, Qc, Ca

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Topological Codes

- Logical subspace $\rightarrow$ linked to the topology of the system
Motivation

Topological Codes

- Logical subspace → linked to the topology of the system
- Operators highly non-local → tailored to resist local noise
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- Operators highly non-local $\rightarrow$ tailored to resist local noise
- Error correction requires local measurements and operations
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- Kitaev’s toric code $\rightarrow$ useful toy model
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- Operators highly non-local $\rightarrow$ tailored to resist local noise
- Error correction requires local measurements and operations
- Kitaev’s toric code $\rightarrow$ useful toy model
- Quantum error-correction (QEC) $\rightarrow$ fast decoding algorithms
1. Kitaev’s Toric Code

2. Concatenation

3. Topological Codes Decoding
1 Kitaev’s Toric Code
   - Stabilizer generators
   - Logical Operators
   - Topology

2 Concatenation

3 Topological Codes Decoding
Stabilizer Generators
Lattice

- 2D square lattice
- Periodic boundary conditions
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Lattice + Qubits

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- A qubit per edge
Kitaev's Toric Code

Lattice + Qubits

- 2D square lattice
- Periodic boundary conditions
- A qubit per edge
- $\Rightarrow 2\ell^2$ qubits
Stabilizer Generators

- Site (vertex) operator: 
  \[ A_s = \prod_{i \in v(s)} X_i \]
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- Plaquette operator: 
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Stabilizer Generators

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- Plaquette operator: \[ B_p = \prod_{i \in v(p)} Z_i \]
- \( \ell^2 \) site and plaquette operators
Stabilizer Generators

\[ [A_s, A_{s'}] = [B_p, B_{p'}] = 0 \]
Stabilizer Generators

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Stabilizer Generators

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- The code is spanned by the simultaneous +1 eigenstates of all these

$$C = \{ |\psi\rangle : A_s |\psi\rangle = |\psi\rangle, B_p |\psi\rangle = |\psi\rangle (\forall s, p) \}$$
Stabilizer Generators

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Stabilizer Generators

\[ \prod_s A_s = I \]
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Stabilizer Generators

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\[ \Rightarrow 2\ell^2 - 2 \text{ independent generators} \]
Stabilizer Generators

- $\prod_s A_s = I$
- $\prod_p B_p = I$
- $\Rightarrow 2\ell^2 - 2$ independent generators
- $\Rightarrow 2$ logical qubits
Logical Operators
First Logical Qubit

$$\overline{Z}_1 = \prod_{i \in \gamma_1} Z_i$$
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Kitaev's Toric Code

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- \( \{ \overline{X}_1, \overline{Z}_1 \} = 0 \)
Second Logical Qubit

- By reflecting around the diagonal
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By reflecting around the diagonal

- \( \{X_2, Z_2\} = 0 \)
- \( [X_2, Z_1] = 0 \)
- \( [X_1, Z_2] = 0 \)
- \( [X_1, X_2] = 0 \)
- \( [Z_1, Z_2] = 0 \)
New Basis

\[ A_1, \ldots, A_{n/2 - 1}, B_1, \ldots, B_{n/2 - 1}, \bar{Z}_1, \bar{Z}_2 \]

\[ tA_1, \ldots, tA_{n/2 - 1}, tB_1, \ldots, tB_{n/2 - 1}, \bar{X}_1, \bar{X}_2 \]
Topology
Trivial Cycles
Trivial Cycles
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All $A_s$, $B_p$ are trivial cycles

Topologically and logically trivial
Trivial Cycles

- All $A_s$, $B_p$ are trivial cycles
- They act as the identity on the code space:
  
  $$A_s |\psi\rangle = B_p |\psi\rangle = +1 |\psi\rangle$$
Trivial Cycles

- All $A_s, B_p$ are trivial cycles
- They act as the identity on the code space:
  \[ A_s \ket{\psi} = B_p \ket{\psi} = +1 \ket{\psi} \]
- Topologically and logically trivial
\( \{ A_s, B_p \} \) span the set of trivial cycles
Trivial Cycles

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Trivial Cycles

- $\{A_s, B_p\}$ span the set of trivial cycles
- $\Rightarrow$ all trivial cycles are equivalent to the identity on the code space
Non-Trivial Cycles
Non-Trivial Cycles
Non-Trivial Cycles
Non-Trivial Cycles
Non-Trivial Cycles

- $\mathbb{Z}_1$ and $\mathbb{Z}_2$ wind around the torus: non-trivial cycles
- They live on the lattice
Non-Trivial Cycles

- $\overline{X}_1$ and $\overline{X}_2$ are conjugate to $\overline{Z}_1$ and $\overline{Z}_2$
- They live on the dual lattice
Non-Trivial Cycles

- Non-trivial cycles have non-trivial effects on the code space
Homological/Logical Classes

\[ |\psi\rangle = B_{p'} |\psi\rangle \]
Homological/Logical Classes

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- $\overline{Z}_1 |\psi\rangle = \overline{Z}_1 B_{p'} |\psi\rangle$
Homological/Logical Classes

- $|\psi\rangle = B_p' |\psi\rangle$
- $Z_1 |\psi\rangle = Z_1 B_p' |\psi\rangle$
- $Z_1 = Z_1 B_p'$
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- $\overline{Z}_1 \equiv \overline{Z}_1 B_{p'} B_{p''}$
Kitaev’s Toric Code

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\[ \overline{Z}_1 \equiv \overline{Z}_1 B_{p'} \]
\[ \overline{Z}_1 \equiv \overline{Z}_1 B_{p'} B_{p''} \]
\[ \overline{Z}_1 \equiv \overline{Z}_1 \prod_p B_p \]
Summary

Stabilizer ↔ Topology

- Every element of the stabilizer is a trivial cycle and vice-versa
- Every logical operator is a non-trivial cycle and vice-versa
- $\Rightarrow$ Topological equivalence classes
1. Kitaev’s Toric Code

2. Concatenation
   - Concatenated codes
   - Efficient Optimal Decoder

3. Topological Codes Decoding
Concatenated Codes
Fast Decoders for Topological Quantum Codes

Codes

\[ |\psi\rangle \rightarrow \left[ [n,k,d] \right] \rightarrow |\overline{\psi}\rangle \]

Concatenation

Concatenated codes
Concatenated Codes

- $k$ layers of encoding $\rightarrow n^k$ qubits
- Error rate decays doubly exponentially: $k \sim \log \epsilon$
Efficient Optimal Decoder
Optimal (Soft) Decoder

\[ \{ \mathcal{P}(I), \mathcal{P}(X), \mathcal{P}(Y), \mathcal{P}(Z) \} \]

- Exponential in \( n \), but \( n \) is constant
- Distillates error probability on the logical qubits
Recursive Decoder

- $k$ layers with at most $n^k$ codes
- Complexity: $O(n^k k)$
1. Kitaev’s Toric Code

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3. Topological Codes Decoding
The threshold is the noise strength under which it is useful to encode.
Previous Method

- PMA : perfect matching algorithm (Preskill, Landahl et al.)
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- Enabled decoding of a $\ell = 1024$ lattice without parallelizing
- More resilient to noise: threshold of $\sim 16.5\%$ under depolarizing noise
- Not limited to toric code (e.g. color codes: triplet of defects)
Subcode

Imagine we had a surface encoding taking 2 qubits into 8.
Toric Code: A concatenation?

- We could recurse on this encoding to build a bigger surface code.
Toric Code : A concatenation?

- We could recurse on this encoding to build a bigger surface code.
If the toric code is just a concatenated code, then we know how to decode it efficiently!
Incomplete Stabilizers

- Some of the stabilizers are incomplete
We complete the stabilizer by adding qubits to the subcode.
By adding these qubits the construction is no more a concatenation.
Even though shared qubits correspond to the same physical entity, we are going to treat them as two different qubits with the same noise model.

Main approximation: Decode with the concatenated code decoder anyway.
Characterizing the subcode

- SubCode stabilizer generators : 10
Characterizing the subcode

- SubCode stabilizer generators : 10
- ⇒ 2 logical qubits
Results

- Is there a threshold at all? At best, these are size effects
By treating shared qubits as independent ones, we introduce inconsistencies.

A compromise between this and exact decoding would be to enforce consistency.
Generalized Belief Propagation (Jonathan S. Yedidja)

- Self-consistency constraints on shared qubits
- Neighboring unit cells exchange messages → Belief propagation
- Compromise on shared qubits
Intuition about GBP

Depolarizing Channel, $p << 1$
Intuition about GBP

Depolarizing Channel

$P(X) \sim 50\%$

$P(X) \sim p^2$

$P(X) \sim 50\%$

$P(X) \sim 50\%$
Intuition about GBP

0

1

2

3

$P(X) \sim p^2$ $P(X) \sim 50\%$

$P(X) \sim 50\%$ $P(X) \sim 50\%$
Intuition about GBP

$P(X) \sim \frac{1}{2}$

$P(X) \sim p^2$
Results

![Graph showing probability of error versus depolarizing channel strength for different lattice sizes.]

- $l=8$
- $l=16$
- $l=32$
- $l=64$

The graph plots the probability of error of the decoder against the depolarizing channel strength ($p$) for various lattice sizes ($l$).
Preliminary Physical Decoding

- BP on the bare stabilizers and qubits
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- Accounts correlations between $X$ and $Z$ introduced by $Y$
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- BP on the bare stabilizers and qubits
- Accounts correlations between $X$ and $Z$ introduced by $Y$
- Its output is the input to the concatenated decoder
Results

![Graph showing the results of error probability as a function of depolarizing channel strength for different code lengths.](image)

- **Probaibilité d’erreur du décodeur**
- **Force du canal depolarizant, p (%)**
- **PMA**

The graph illustrates the error probability for decoders on topological quantum codes with varying code lengths, indicating how error rates increase with increasing channel strength.
Unit cell \((2 \times 1)\) + decoding the 2 types of defects independently \(\Rightarrow \ell = 1024\) lattice
Conclusion

- Topological codes use highly non-local operators to encode information
- We proposed an efficient ($O(\log \ell)$ time) to decode them → Concatenated codes, GBP
- More resilient to noise under depolarizing noise than known methods (16.5% vs. 15.5%)
- It enabled decoding of color codes $p_{th} \sim 8.7\%$ (Héctor Bombin)
Work in progress: Color Codes
Color Codes: Mapping

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Color Codes: Results

Decoding error probability against Bit-Flip channel strength $p\%$ for different values of $l$ (labeled as $l=16, 32, 64, 128, 256$).